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Method for Solving Simultaneous Helmholtz Equations

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ABSTRACT—The method presented is based on a similarity transformation of the coefficients matrix allowing the transformed variables to be relaxed independently in a recursive fashion.

1. INTRODUCTION

Semi-implicit time integrations are currently attracting greater interest in modeling of atmospheric problems. Some numerical consequences of this approach are the introduction of Helmholtz-type equations into the predictive scheme. The manner in which these boundary value problems arise depends on the choice of terms to be treated implicitly; but regardless of this choice, in multilevel formulations, the resulting Helmholtz equations are coupled.

In principle, it is possible to solve a system of Helmholtz equations by simultaneous relaxation. This method is neither efficient nor elegant. It is our purpose to describe a method that effectively decouples the system. Limited experiments with a two-layer model, performed at the National Meterological Center (NMC) by McPherson (1972), indicate roughly double the efficiency of a simultaneous ordinary relaxation. The success of applying the method to a 10-layer scheme currently under development at NMC lends this method further interest.

2. DERIVATION

We are considering a system of Helmholtz equations,

$$\nabla^2\mathbf{Y}+\mathbf{CY}=\mathbf{F} \quad (\text{or } \nabla^2\mathbf{C}'\mathbf{Y}+\mathbf{Y}=\mathbf{F}') \tag{1}$$

where $\mathbf{C}$ is a time-independent square matrix of order equal to the number of layers in the atmospheric model, $\mathbf{F}$ is a column vector, and $\mathbf{Y}$ is the column of dependent variables. Either form of eq (1) can be obtained from the original set of equations.

We consider the first equation and rewrite it in the following way:

$$\nabla^2(\mathbf{BY})+\mathbf{BCB}^{-1}(\mathbf{BY})=\mathbf{BF} \tag{2}$$

where $\mathbf{B}$ is a regular matrix. If, in particular, $\mathbf{B}$ is such that $\mathbf{T}=\mathbf{BCB}^{-1}$ is triangular, then, for the transformed variable $\mathbf{W}$,

$$\mathbf{W}=\mathbf{BY}, \tag{3}$$

we have

$$\nabla^2\mathbf{W}+\mathbf{TW}=\mathbf{BF} \tag{4}$$

and eq (4) is a recursive system of simultaneous equations.

We first solve for W_1 from

$$\nabla^2W_1+T_{11}W_1=B_{11}F_1 \tag{5}$$

and proceed to solve, in general,

$$\nabla^2W_k+T_{kk}W_k=\sum_j B_{kj}F_j-\sum_{j=1}^{k-1} T_{kj}W_j. \tag{6}$$

Once the Ws are found, the original dependent variables can be computed by inverting eq (3). The major problem is to find the transformation matrix, $\mathbf{B}$.

The following algorithm is cited (Wilkinson 1965). Let

$$\begin{gathered}\mathbf{C}=\mathbf{L}_1\mathbf{U}_1\\ \mathbf{L}_2\mathbf{U}_2=\mathbf{U}_1\mathbf{L}_1\\ \vdots\\ \mathbf{L}_k\mathbf{U}_k=\mathbf{U}_{k-1}\mathbf{L}_{k-1}\end{gathered}$$

where the $\mathbf{L}$s and $\mathbf{U}$s are lower and upper triangular matrices and the $\mathbf{U}$s have a unit diagonal. Then

$$\mathbf{C}=\mathbf{L}_1\mathbf{U}_1=\mathbf{U}_1^{-1}\mathbf{L}_2\mathbf{U}_2\mathbf{U}_1=\mathbf{U}_1^{-1}\mathbf{U}_2^{-1}\mathbf{L}_3\mathbf{U}_3\mathbf{U}_2\mathbf{U}_1\cdots$$

$$\mathbf{C}=\mathbf{U}_1^{-1}\mathbf{U}_2^{-1}\cdots\mathbf{U}_{k-1}^{-1}(\mathbf{L}_k\mathbf{U}_k)\mathbf{U}_{k-1}\mathbf{U}_{k-2}\cdots\mathbf{U}_1;$$

and, under appropriate conditions (which unfortunately cannot, in general, be examined and are related to factorization of the characteristic polynomial of $\mathbf{C}$ into simple binomial terms), the matrix $\mathbf{L}_k\mathbf{U}_k$ will approach a triangular form. If we define

$$\mathbf{B}=\mathbf{U}_{k-1}\mathbf{U}_{k-2}\cdots\mathbf{U}_1,$$

then

$$\mathbf{L}_k\mathbf{U}_k=\mathbf{BCB}^{-1}$$

and the matrix $\mathbf{B}$ has to be constructed in steps until $\mathbf{L}_k\mathbf{U}_k$ is "sufficiently triangular."

As an example of the method, a 10×10 matrix arising from the 10-layer model previously mentioned is treated by this algorithm. Figure 1 shows the given matrix, $\mathbf{C}$, figure 2 displays the transformation matrix, $\mathbf{B}$, and figure 3 portrays the triangular matrix, $\mathbf{BCB}^{-1}$. The results indicate very good accuracy and were checked with single and double precision.

−0.41D+00	−0.27D+05	0.22D+05	−0.18D+05	0.14D+05	−0.98D+04	0.66D+04	−0.40D+04	0.19D+04	−0.59D+03
− .28D−03	.17D+04	− .20D+04	.16D+04	− .12D+04	.87D+03	− .59D+03	.35D+03	− .17D+03	.53D+02
− .15D−03	− .17D+04	.27D+04	− .26D+04	.20D+04	− .15D+04	.98D+03	− .59D+03	.28D+03	− .88D+02
− .99D−04	.11D+04	− .22D+04	.27D+04	− .24D+04	.18D+04	− .12D+04	.71D+03	− .34D+03	.11D+03
− .20D−03	− .71D+03	.14D+04	− .20D+04	.22D+04	− .18D+04	.12D+04	− .75D+03	.36D+03	− .11D+03
− .15D−04	.40D+03	− .77D+03	.11D+04	− .14D+04	.15D+04	− .12D+04	.70D+03	− .33D+03	.10D+03
− .16D−03	− .21D+03	.39D+03	− .56D+03	.71D+03	− .85D+03	.84D+03	− .59D+03	.28D+03	− .88D+02
.66D−05	.80D+02	− .15D+03	.22D+03	− .28D+03	.34D+03	− .39D+03	.36D+03	− .21D+03	.66D+02
− .80D−04	− .26D+02	.44D+02	− .60D+02	.75D+02	− .88D+02	.10D+03	− .11D+03	.91D+02	− .40D+02
− .87D−06	.47D+01	− .92D+01	.13D+02	− .17D+02	.20D+02	− .23D+02	.25D+02	− .27D+02	.21D+02

FIGURE 1.—The given matrix, C, from the 10-layer model.

NU=50

0.10D+01	0.26D+08	−0.43D+08	0.50D+08	−0.47D+08	0.39D+08	−0.28D+08	0.18D+08	−0.85D+07	0.27D+07
0	.10D+01	− .16D+01	.17D+01	− .15D+01	.12D+01	− .81D+00	.49D+00	− .24D+00	.74D−01
0	0	.10D+01	− .19D+01	.16D+01	− .47D+00	− .32D+00	.49D+00	− .32D+00	.11D+00
0	0	0	.10D+01	− .24D+01	.27D+01	− .19D+01	.94D+00	− .37D+00	.10D+00
0	0	0	0	.10D+01	− .20D+01	.13D+01	− .27D−01	− .39D+00	.20D+00
0	0	0	0	0	.10D+01	− .22D+01	.20D+01	− .92D+00	.23D+00
0	0	0	0	0	0	.10D+01	− .34D+00	− .19D+01	.18D+01
0	0	0	0	0	0	0	.10D+01	− .18D+01	.10D+01
0	0	0	0	0	0	0	0	.10D+01	− .18D+01
0	0	0	0	0	0	0	0	0	.10D+01

FIGURE 2.—The transformation matrix, **B**, from the 10-layer model.

NU=50

0.89D+04	0.17D−15	−0.21D−15	0.15D−17	−0.11D−15	0.35D−15	0.58D−15	−0.23D−15	0.97D−15	0.86D−15
.24D−03	.19D+04	− .28D−18	.17D−24	− .17D−23	.12D−22	.19D−22	− .92D−23	.28D−22	.28D−22
− .19D−03	− .17D+03	.72D+03	− .22D−14	.20D−24	− .90D−23	− .13D−22	.82D−23	− .24D−22	− .20D−22
.69D−03	− .13D+05	.46D+02	.31D+03	− .62D−14	.24D−22	.48D−22	− .22D−22	.74D−22	.63D−22
− .35D−03	.73D+04	− .19D+03	− .16D+03	.14D+03	− .86D−15	− .23D−22	.13D−22	− .37D−22	− .30D−22
.44D−03	− .10D+05	.62D+03	− .30D+03	.28D+01	.62D+02	− .11D−16	− .14D−22	.44D−22	.36D−22
− .20D−04	.33D+03	.17D+02	− .44D+02	.63D+02	− .12D+03	.26D+02	− .40D−18	− .17D−23	− .14D−23
.15D−03	− .36D+04	.28D+03	− .20D+03	.11D+03	− .90D+02	.59D+01	.98D+01	.15D−22	.12D−22
− .78D−04	.20D+04	− .16D+03	.12D+03	− .73D+02	.75D+02	− .35D+02	.54D+02	.20D+01	− .66D−23
− .87D−06	.27D+02	− .35D+01	.35D+01	− .32D+01	.49D+01	− .51D+01	.14D+02	− .88D+01	− .48D+00

FIGURE 3.—The triangular matrix, $\mathbf{BCB}^{-1}$, from the 10-layer model.

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PICTURE OF THE MONTH

Ship Trails or Anomalous Cloud Lines

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Numerous cases of anomalous lines have appeared in satellite pictures. These lines have been observed in both the Atlantic and the Pacific Oceans, but most frequently appear off the California coast in late spring and early summer.

Conover (1966, 1969) presented an interesting discussion on the cause or origin of anomalous lines in satellite photographs. In 1967, Weather Bureau Western Region offices, with the cooperation of commercial airline pilots, began to investigate these lines.

One documented case is shown here. On June 27, 1967, the ESSA 2, automatic picture transmission (APT) photograph (fig. 1), taken at 1720 GMT, showed anomalous cloud lines (upper right-hand corner). B. J. Haley, pilot of United Air Lines flight 185–27, flew over this area at 1945 GMT. He reported no clouds or vapor trails at his flight level of 31,000 ft. Below him, he observed a layer of low, thin stratus through which sunlight, reflected from the ocean, could be seen. He estimated the height of the stratus layer to be 1,000 ft above the surface. Haley also re-

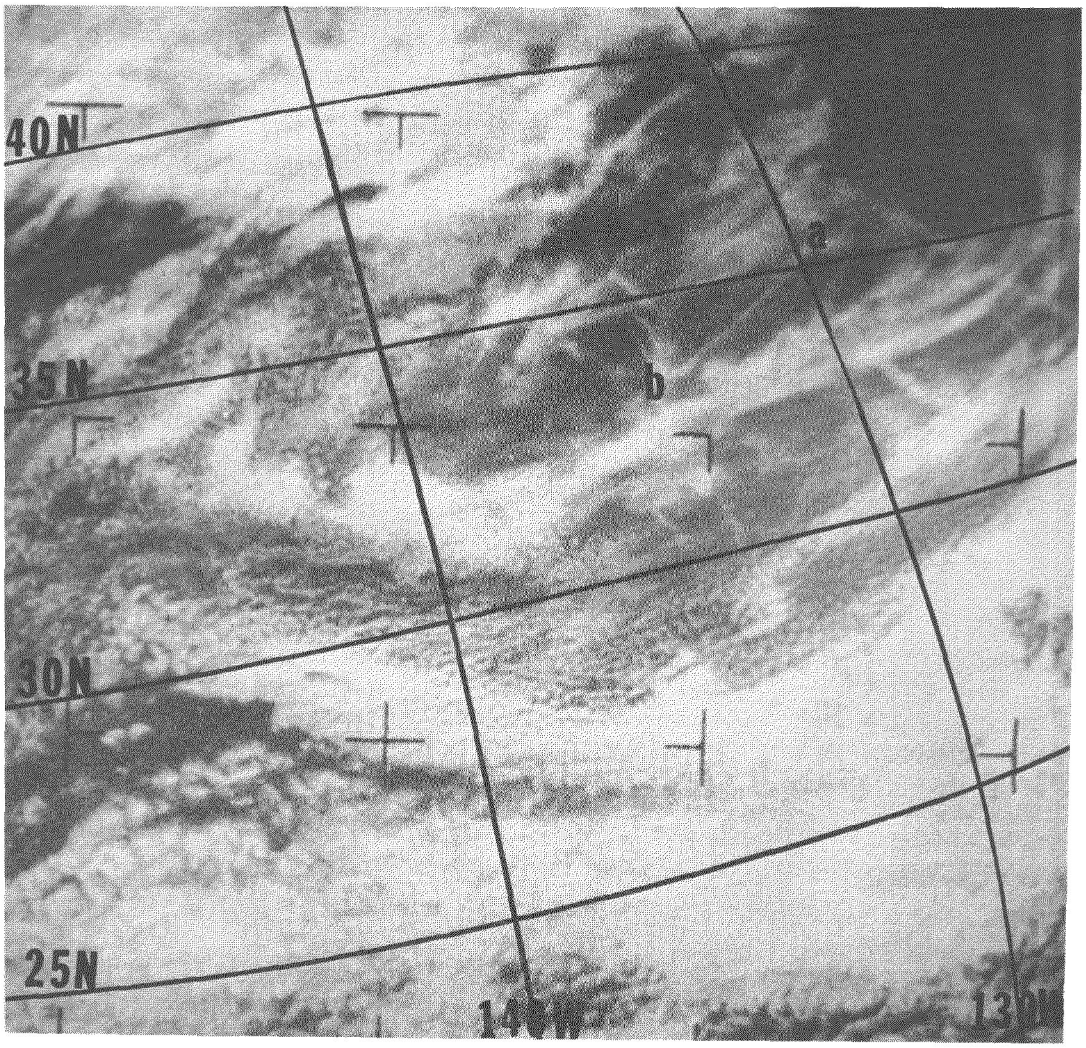

FIGURE 1.—ESSA 2, APT, on June 27, 1967, 1720 GMT.

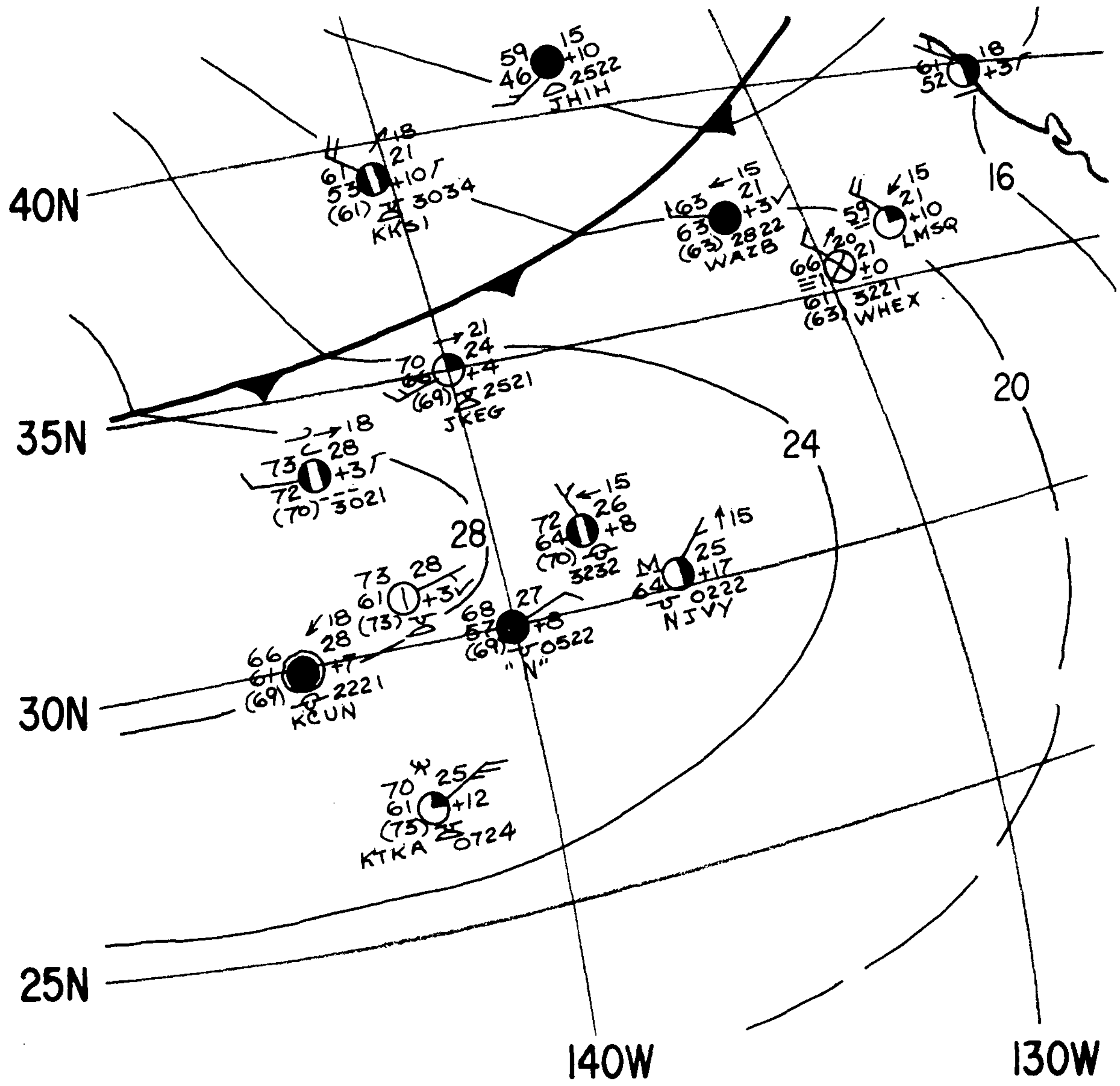

FIGURE 2.—Surface map for 1800 GMT, June 27, 1967.

ported that a ship located near 35°N, 131°W and traveling due east was clearly visible through the stratus layer. The smoke from the ship's stack was forming a "definite line of thicker clouds" in the layer of scattered and broken stratus. This condensation trail was estimated to be 2–4-mi wide at a point 10 mi behind the ship, and to extend 125–150 mi before fading out. It was drifting southward with the cloud layer.

This observed condensation trail had the same orientation as a line (a–b) in figure 1.[1] The 1800 GMT surface reports shown in figure 2 indicate that two ships, WHEX and WAZB, were in this area. Ship WHEX was located to the east of the cloud line and was traveling northeastward at 20 kt. Ship WAZB was traveling westward at 15 kt away from this area.

[1] The gridding error of an APT picture without landmarks in its central portion can be as large as ± 1° latitude.

The surface analysis (fig. 2) shows these ships to be located in the eastern sector of a large high-pressure area. At this time, 5–10-kt northwesterly winds, fog, and stratus, typical of the ship trail-producing regime, were reported.

Recent observations of these anomalous lines in the Applications Technology Satellite 1 data show that they are distorted and propagated by low-level winds. Once the synoptic regime becomes established, these lines will appear in various configurations for 2 or 3 successive days.

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